Conclusions

The paper has studied the optimal low-thrust Earth–moon targeting strategy for the n-body problem. First, the optimal, low-thrust reference Earth–moon trajectory is established. Then, taking the osculating orbital elements as the independent variables and adjusting the optimal low-thrust, reference Earth–moon trajectory with the differential correction algorithm, we achieve the targeting parameters. The numerical results confirm the algorithm works successfully.

Acknowledgment

This work was funded by the Chinese National Science Foundation.

References

[1]Kluever, C. A., and Pierson, B. L., "Optimal Low-Thrust Three-Dimensional Earth–Moon Trajectories," *Journal of Guidance, Control, and Dynamics*, Vol. 18, No. 4, 1995, pp. 830–837.

[2]Herman, A. L., and Conway, B. A., "Optimal, Low-Thrust, Earth–Moon Orbit Transfer," *Journal of Guidance, Control, and Dynamics*, Vol. 21, No. 1, 1998, pp. 141–147.

[3]Yan, H., and Wu, H., "Initial Adjoint-Variable Guess Technique and Its Application in Optimal Orbital Transfer," *Journal of Guidance, Control, and Dynamics*, Vol. 22, No. 3, 1999, pp. 490–492.

[4]Bate, R. R., Mueller, D. D., and White, J. E., *Fundamentals of Astrodynamics*, Dover, New York, 1971, pp. 80–83.

[5]Carrico, J., Hooper, H. L., Roszman, L., and Gramling, C., "Rapid Design of Gravity Assist Trajectories," *Proceedings of the ESA Symposium on Spacecraft Flight Dynamics*, European Space Agency, Darmstadt, Germany, 1991, pp. 427–434.

[6]Folta, D. C., and Sharer, P. J., "Multiple Lunar Flyby Targeting for the WIND Mission," *Spaceflight Mechanics 1996*, Vol. 93, Advances in the Astronautical Sciences, Univelt, San Diego, CA, 1996, pp. 53–67.

Range, Range Rate, and Acceleration Computation for Inclined Geosynchronous Orbit

Kar W. Yung* and Peter B. Landecker*
Hughes Space and Communications Company
Los Angeles, California 90009

Nomenclature

i = inclination angle of the geosynchronous satellite
$\boldsymbol{R}$ = position vector of the geosynchronous satellite with respect to a ground location
$\boldsymbol{R}_G$ = position vector of the ground location relative to the center of the Earth
$\boldsymbol{R}_S$ = position vector of the geosynchronous satellite relative to the center of the Earth
$\dot{\boldsymbol{R}}$ = velocity vector of the geosynchronous satellite with respect to a ground location
$\dot{\boldsymbol{R}}_S$ = velocity vector of the geosynchronous satellite
r = range of the geosynchronous satellite with respect to a ground location
r_G = radius of Earth, approximately 6378 km
r_S = geosynchronous orbital radius, approximately 42,165 km
$\dot{r}$ = range rate of the geosynchronous satellite with respect to a ground location
$\dot{r}_{\max}$ = maximum range rate for all possible ground locations within the field of view of the satellite
$\dot{r}_{\max 1}$ = maximum range rate for one single ground location
t = time elapsed since passage of satellite from the ascending node
η = latitude
θ = central angle between satellite and ground unit position vectors
λ = longitudinal displacement from the ascending node
ω = angular rate of Earth as well as geosynchronous orbit

Received 3 December 1999; revision received 31 July 2000; accepted for publication 1 November 2000. Copyright © 2001 by the American Institute of Aeronautics and Astronautics, Inc. All rights reserved.

*Senior Scientist, System Products, P.O. Box 92919.

Introduction

THIS Note derives closed-form solutions for range, range rate, and range acceleration between an inclined geosynchronous satellite and locations on the ground surface. The motivation of this exercise is to avoid the complexities of numerical analyses (for example, in FORTRAN) by instead using a simple algebraic formula solution amenable for easy use (for example, in spreadsheets). A practical reason for why this calculation is useful is the need to know the maximum Doppler shift for devices for communication gateways as part of the filter bandpass design specification.

The coordinate system is defined. Each ground location is specified by its latitude and longitudinal displacement from the ascending node. Closed-form equations for range, range rate, and range acceleration are then given. An expression for maximum possible range rate is also derived. The resulting solutions are simple algebriac expressions.

These results are potentially useful for several commercial satellite communication programs that employ constellations of inclined geosynchronous Earth orbit (GEO) satellites. This calculation has not been previously published[1−3] (Wertz, J. R., private communication).

In our model, we assume the Earth is a perfect sphere and it rotates at the same angular rate as the Earth. The vectors $\boldsymbol{R}$, $\dot{\boldsymbol{R}}$, $\boldsymbol{R}_G$, $\boldsymbol{R}_S$, and $\dot{\boldsymbol{R}}_S$ are defined in a coordinate system that rotates with the Earth and are shown in Fig. 1. Its x axis points toward the ascending node and its z axis points toward the north pole.

Range and Its Derivatives

The position vectors of the ground location and the satellite are as follows:

$$\boldsymbol{R}_G = r_G \begin{pmatrix} \cos\lambda\cos\eta \\ \sin\lambda\cos\eta \\ \sin\eta \end{pmatrix} \tag{1}$$

$$\boldsymbol{R}_S = r_S \begin{pmatrix} \frac{1}{2}\{1+\cos(i)+[1-\cos(i)]\cos(2\omega t)\} \\ -\frac{1}{2}[1-\cos(i)]\sin(2\omega t) \\ \sin(i)\sin(\omega t) \end{pmatrix} \tag{2}$$

$$\boldsymbol{R} = \boldsymbol{R}_S - \boldsymbol{R}_G \tag{3}$$

The range between the ground location and the satellite is given by

$$r = \sqrt{r_S^2 + r_G^2 - 2 r_S r_G A} = r_S\sqrt{1 + (r_G/r_S)^2 - 2(r_G/r_S)A} \tag{4}$$

where

$$A = \tfrac{1}{2}\cos\lambda\cos\eta[1+\cos(i)] + \sin\eta\sin(i)\sin(\omega t) + \tfrac{1}{2}\cos\eta[1-\cos(i)]\cos(\lambda + 2\omega t) \tag{5}$$

The ratio of ranges is defined as follows:

$$\beta = r_G/r_S \tag{6}$$

Because the satellite is always above the surface of the Earth,

$$0 < \beta < 1 \tag{7}$$

The range is, therefore, given by

$$r = r_S\sqrt{1 + \beta^2 - 2\beta A} \tag{8}$$

The velocity vector of the satellite is given by

$$\dot{\boldsymbol{R}} = \dot{\boldsymbol{R}}_S = r_S\omega \begin{pmatrix} -\frac{1}{2}[1-\cos(i)]\sin(2\omega t) \\ \frac{1}{2}\{1+\cos(i)-[1-\cos(i)]\cos(2\omega t)\} \\ \sin(i)\cos(\omega t) \end{pmatrix} \tag{9}$$

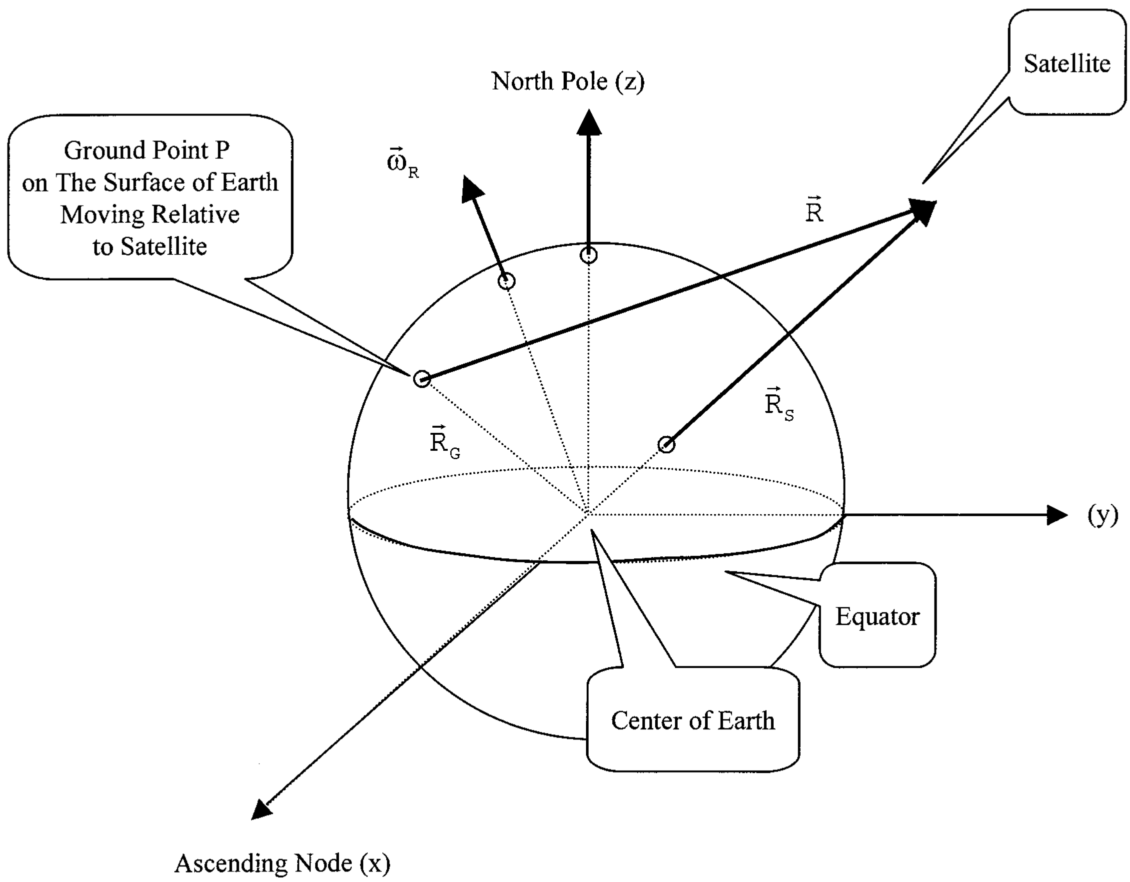

Fig. 1 Position vectors of the satellite and ground; ground point P on the surface of the Earth moves toward the satellite with relative angular velocity ω_R.

The factors $\frac{1}{2}$ are there because $\boldsymbol{R}_S$ is not an inertial vector. Its derivative $\dot{\boldsymbol{R}}_S$ is the time derivative plus the cross product of the Earth's angular velocity $\boldsymbol{\omega}_G$ and $\boldsymbol{R}_S$.

The range rate between the ground location and the satellite is

$$\dot{r} = \frac{\boldsymbol{R}\cdot\dot{\boldsymbol{R}}}{r} = \frac{r_S r_G \omega}{r} B = r_G \omega \frac{B}{\sqrt{1+\beta^2-2\beta A}} \tag{10}$$

where

$$B = -\sin\eta \sin(i)\cos(\omega t) + \cos\eta[1-\cos(i)]\sin(\lambda+2\omega t) \tag{11}$$

Note that $B = -\mathrm{d}A/\mathrm{d}t/\omega$. The range acceleration is given by

$$\ddot{r} = \frac{\omega^2 \beta r_G}{\left(\sqrt{1+\beta^2-2\beta A}\right)^3}\left(\frac{\beta^2+1}{\beta}C - 2AC - B^2\right)$$

$$= \frac{r_S r_G \omega^2 C - \dot{r}^2}{r} \tag{12}$$

where

$$C = \sin(\eta)\sin(i)\sin(\omega t) + 2\cos(\eta)[1-\cos(i)](\cos\lambda + 2\omega t) \tag{13}$$

Note that $C = \mathrm{d}B/\mathrm{d}t/\omega$.

Maximum Range Rate

When the range rate at any geographic location is at its maximum, we must have

$$\ddot{r} = 0 \tag{14}$$

With Eq. (12), we derive

$$[(\beta^2+1)/\beta]C - 2AC - B^2 = 0 \tag{15}$$

Also, when the maximum occurs, the value of this maximum must be

$$\dot{r}_{\max 1} = \omega\sqrt{r_S r_G C} \tag{16}$$

Solving Eq. (16) for the time of maximum range at any location in closed form is hard, but finding the maximum range rate among all ground locations could be relatively easy. To derive this, we have to realize that A is really the cosine of the central angle θ between satellite and ground unit position vectors or dot product of these two unit vectors:

$$A = \cos(\theta) \tag{17}$$

$$B = -(\dot{A}/\omega) = \sin(\theta)(\dot{\theta}/\omega) \tag{18}$$

Therefore,

$$B^2 = (1-A^2)(\dot{\theta}/\omega)^2 \tag{19}$$

Substituting Eq. (19) in Eq. (15), we have

$$C = \frac{(1-A^2)\beta}{1+\beta^2-2\beta A}\left(\frac{\dot{\theta}}{\omega}\right)^2 \tag{20}$$

Equation (20) consists of two factors that are independent variables. The first factor $(1-A^2)\beta/(1+\beta^2-2\beta A)$ is a function of angular separation between ground and satellite vectors. The second factor $(\dot{\theta}/\omega)^2$ is the rate of change of this angular separation. We will treat them independently. We look for separate maxima for these two factors. The maxima do not necessarily have to occur at the same point on the orbit (although they did for all cases where we did a numerical and closed-form solution comparison). In any case, an upper bound to the point of physical maximum is mathematically determined.

We take the derivative with respect to A and set it to zero to determine $C_{\max}$,

$$\left.\frac{\partial C}{\partial A}\right|_{C_{\max}} = 2\beta\frac{(1-A\beta)(\beta-A)}{(1+\beta^2-2\beta A)^2}\left(\frac{\dot{\theta}}{\omega}\right)^2 = 0 \tag{21}$$

There are two solutions. For the satellite above Earth,

$$A = \beta \tag{22}$$

For the satellite below Earth,

$$A = 1/\beta \tag{23}$$

Only the first solution is physically meaningful due to Eq. (7). In fact, for all ground points within the field of view of the satellite, it must be true that $1 \geq A \geq \beta$. When the second factor of Eq. (20) is examined, its maximum value corresponds to the maximum angular rate between the ground and satellite vectors:

$$(\dot{\theta}/\omega)^2 = (\omega_R/\omega)^2 \tag{24}$$

where ω_R is the relative angular rate.

The relative angular velocity of any ground point relative to the satellite is given by

$$\boldsymbol{\omega}_R = \boldsymbol{\omega}_G - \boldsymbol{\omega}_S \tag{25}$$

Now examine the relative angular rate,

$$\omega_R = |\boldsymbol{\omega}_R| = \sqrt{|\boldsymbol{\omega}_G|^2 + |\boldsymbol{\omega}_S|^2 - 2\boldsymbol{\omega}_G \cdot \boldsymbol{\omega}_S} = \omega\sqrt{2[1 - \cos(i)]} \tag{26}$$

Therefore

$$(\dot{\theta}/\omega)^2 = (\omega_R/\omega)^2 = 2[1 - \cos(i)] \tag{27}$$

Substituting Eqs. (22) and (27) in Eq. (20) yields

$$C_{\max} = 2\beta[1 - \cos(i)] \tag{28}$$

Thus, the maximum range rate among all ground points within the field of view of the satellite becomes

$$\dot{r}_{\max} = \omega\sqrt{r_S r_G C_{\max}} = r_G \omega\sqrt{2[1 - \cos(i)]} \cong r_G \omega \sin(i) \quad \text{for} \quad i \ll 1 \tag{29}$$

We can also derive the preceding maximum range rate results with another approach that is less mathematically rigorous, yet more physically insightful, as follows.

The Cartesian position vector $\boldsymbol{R}_G$ of any ground point depends on time, its longitude, and its latitude. It is not a simple function. However, if we look at the kinematics of all ground points from the angular velocity perspective, it is actually very simple. All ground points have the same instantaneous angular speed because they are part of one single rigid body. Yet the axis of this angular velocity oscillates from the satellite's point of view due to its motion.

From Fig. 1, we can see that all ground points within the field of view of the satellite are moving at the angular rate ω_R. There is only one possible point P moving away from the satellite at the maximum speed or range rate $r_G \omega_R$. Therefore, we conclude that

$$\dot{r}_{\max} = r_G \omega_R = r_G \omega\sqrt{2[1 - \cos(i)]} \tag{30}$$

which is the same result as Eq. (29) derived earlier. For the example of a GEO satellite orbit with inclination 5.3 deg, the maximum possible range rate is given by $\dot{r}_{\max} = 0.043$ km/s.

Conclusions

We produced a closed-form algorithm and useful equation that permits easy calculation of the range, range rate, and range acceleration for the case of an inclined GEO satellite. We showed two methods to achieve the same results for finding the maximum range rate.

References

[1]Larson, W. J., and Wertz, J. R., *Space Mission Analysis and Design*, 2nd ed., Microcosm, Torrance, CA, and Kluwer, Dordrecht, The Netherlands, 1992, pp. 93–195.

[2]Nicola, L., and Walters, L. G., "Scalar Differential Expressions for the Geostationary Satellite," *Journal of the British Interplanetary Society*, Vol. 19, Pt. 6, 1963, pp. 241–247.

[3]Kinal, G. V., "The INMARSAT Satellite Radionavigation Test Bed," *NAV 89—Satellite Navigation*, Vol. 43, Royal Inst. of Navigation, London, 1989, pp. 18–25.

Variational Calculus and Approximate Solutions of Algebraic Equations

David G. Hull*
University of Texas at Austin, Austin, Texas 78712

Introduction

THE thrust of this Note is the derivation of the equations that are to be solved for an approximate analytical solution of an algebraic equation involving a small parameter, the algebraic perturbation problem. The usual procedure is to recognize that the solution can be expressed in a power series of the small parameter, substitute the series into the algebraic equation, expand in a Taylor series, and equate the coefficients of the powers of the small parameter to zero. This process leads to the equations for the zeroth-order part of the solution, the first-order part, and, so on.

In association with parameter optimization,[1] it has been shown that the Taylor series process is equivalent to a differential process called variational calculus. A variation is a small but finite displacement, its symbol behaves like a differential, and variations of independent variations are zero whereas variations of dependent variations are not zero. Variational calculus has been applied to the initial value problem of ordinary differential equations.[2] Perturbed paths have been generated by perturbing the initial conditions or a small parameter in the differential equation.

In this Note, variational calculus is established for algebraic equations. The expansion process and the variational process are both used to derive the equations for the zeroth-order part, the first-order part, etc., of the algebraic perturbation problem of Kepler's equation with small eccentricity. The efforts required by both processes are compared.

Although only a scalar equation is considered here, the variational process can be applied to vector equations if indicial notion is employed. Matrix notation can be employed if only the first-order correction to the zeroth-order solution is being sought.

Taylor Series Expansions and Variational Calculus

In this section it is shown that Taylor series expansions can be created on a term by term basis by a differential process called variational calculus. To do so, consider the scalar equation

$$y = F(x) \tag{1}$$

which is shown schematically in Fig. 1. In this formulation, x is the independent variable, and y is the dependent variable. Consider the nominal point x, y and the perturbed or neighboring point x_*, y_*. If the symbol Δ denotes a small but finite change, a displacement, the coordinates of the perturbed point are $x_* = x + \Delta x$ and $y_* = y + \Delta y$. For a given Δx, it is desired to find Δy. From $y_* = F(x_*)$, it is seen that

$$y + \Delta y = F(x + \Delta x) \tag{2}$$

Received 26 June 2000; revision received 13 December 2000; accepted for publication 13 December 2000. Copyright © 2001 by David G. Hull. Published by the American Institute of Aeronautics and Astronautics, Inc., with permission.

*M. J. Thompson Regents Professor, Department of Aerospace Engineering and Engineering Mechanics.